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LINEAR SYSTEMS ON $K3$-SECTIONS

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1. Introduction

The types of special linear systems which exist on a curve C which is a hyperplane section of a $K3$ surface X often do not depend on C but only on its linear equivalence class in X. For instance, Saint-Donat proved in [14] that C possesses a g^1_2 or g^1_3 if and only if the same is true for every nonsingular curve $C' \in |C|$, where $|C|$ denotes the linear system of C on X, and Reid [12] found some extensions of this result to other g^1_d's. The general question of whether the presence of a special g^r_d on a given hyperplane section C of a $K3$ surface forces the existence of such a g^r_d on every nonsingular $C' \in |C|$ arose out of work of Harris and Mumford [7]. Our purpose is to study this question and some related conjectures. We use the term $K3$-*section* to denote a smooth curve of genus at least two on a $K3$ surface. (Such a curve, if nonhyperelliptic, is a hyperplane section of a birational model of the $K3$ surface X in some projective embedding.)

We start, in §2, with a counterexample: a $K3$ surface X in $\mathbf{P}^{10}$, some of whose hyperplane sections (but not all) possess a g^1_4. In §3 we use a counting argument to show that if C carries a g^1_d which is scheme-theoretically isolated in moduli, then this g^1_d "propagates" to every nonsingular $C' \in |C|$, in the sense that an explicit geometric construction starting from the g^1_d on C produces a g^1_d on C'. A sufficient condition for the propagation of g^r_d's is also obtained, but it is weak for $r > 1$.

Analysis of our counterexample shows that in the family of all nonsingular hyperplane sections of X, the subfamily of curves carrying a g^1_4 has codimension one. On the other hand, all these curves *do* carry a g^2_6. Combining this observation with his theory of Koszul cohomology, Mark Green suggested that the correct conjecture is not propagation of g^r_d's but constancy of the "Clifford index" $\nu = d - 2r$. More precisely, for a line bundle M on a $K3$-section C with $h^0(M) = r + 1$, $\deg(M) = d$, and $\text{genus}(C) = g$, define

$$\nu(M) := d - 2r, \qquad \nu(C) := \min\{\nu(M) \mid r \geq 1,\ d \leq g - 1\}.$$

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Clifford's theorem says that $\nu(C) \geq 0$, with equality if and only if C is hyperelliptic. We also define

$$\nu(\mathscr{O}_X(C)) := \nu(C') \quad \text{for generic } C' \in |C|.$$

(Notice that the function $C' \mapsto \nu(C')$ is lower semicontinuous on the family of nonsingular curves $C' \in |C|$, so that $\nu(\mathscr{O}_X(C))$ can be characterized as the smallest integer ν such that for every nonsingular $C' \in |C|$ there is some line bundle M' on C' with $h^0(M') \geq 2$, $\deg(M') \leq g-1$ and $\nu(M') \leq \nu$.) Green's conjecture is then:

(1.1) Conjecture [3]. *If X is a $K3$ surface and L is an ample line bundle on X then $\nu(C) = \nu(L)$ for all nonsingular $C \in |L|$.*

In §4 we prove this conjecture for g^1_d's. That is, we show that if the Clifford index of a nonsingular C is achieved by a g^1_d, i.e., if there is a g^1_d on C with $d-2 = \nu(C)$, then $\nu(C) = \nu(\mathscr{O}_X(C))$. Reid [12] had earlier shown this when g is sufficiently large with respect to d.

Another interesting feature of our counterexample is that the g^2_6 linear systems on all the hyperplane sections $C' \in |C|$ are restrictions of one and the same line bundle on X; the same holds for the g^1_2's and g^1_3's studied by Saint-Donat. In a second counterexample, based on an example of Reid [12], we exhibit a $K3$ surface X with an ample linear system $|C|$ such that every $C' \in |C|$ has a g^1_6, but these are not all induced from the same bundle on X. (For generic $C' \in |C|$, these g^1_6's are scheme-theoretically isolated in moduli and have negative Brill-Noether number $\rho < 0$, but are not unique.) Again, each of these g^1_6's is contained in a g^2_8 (which the reader should notice has the same Clifford index $\nu = 4$), and these g^2_8's are induced from a bundle on X. We suggest that this is a general phenomenon:

(1.2) Conjecture. *Let X be a $K3$ surface, C be a smooth curve on X of genus $g \geq 2$, and $|Z|$ be a complete base point free g^r_d on C with $r \geq 1$, $d \leq g-1$, such that*

$$\rho(Z) := (d-r)(r+1) - rg < 0.$$

Then the linear system $|Z|$ is contained in the restriction to C of a linear system $|D|$ on X with

$$\deg(D \cap C) \leq g-1, \qquad \nu(D \cap C) \leq \nu(Z).$$

(We recall that a linear system $|Z|$ on C is *contained in* another system $|Z'|$ if every divisor $Z \in |Z|$ is contained in some $Z' \in |Z'|$, i.e., $Z \leq Z'$ as divisors on C.)

Conjecture (1.2) clearly implies (1.1); this requires an easy computation which we leave to the reader. In §5 we extend the analysis of §4, proving (1.2) for $r = 1$. Once again, the first results in this direction are due to Reid [12], who used Ramanujam's theory of numerical connectedness of divisors on a surface [11]. Our technique in §§4 and 5 is somewhat different: inspired by work of Lazarsfeld [8] and Reider [13], we construct a rank two vector bundle on X in order to study the g^1_d $|Z|$.

After this work had been completed (but before this paper was finished), we received a preprint from Green and Lazarsfeld [4], which proves Green's conjecture (1.1) in full generality, and also a part of (1.2): there is a linear system $|D|$ on X such that $\nu(\mathscr{O}_C(D)) = \nu(C)$. From that preprint we also learned of some work of Tyurin [15] related to our construction in §3.

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2. Linear systems on $K3$ surfaces: review and counterexamples

(2.1) We gather here some useful facts about linear systems on a $K3$ surface X, taken from Mayer [9] and Saint-Donat [14]. To start, we list some examples of exceptional behavior:

$X1$. Let $F \subset X$ be a smooth elliptic curve, and consider $L := \mathscr{O}(kF)$, $k \geq 1$. We then have

$$h^0(L) = k+1, \qquad h^1(L) = k-1,$$

and the map $\varphi_{|L|}$ determined by sections of L sends X to a rational normal curve in $\mathbf{P}^k$. In particular, all divisors in $|L|$ are of the form $\sum_{i=1}^k F_i$ with $F_i \sim F$.

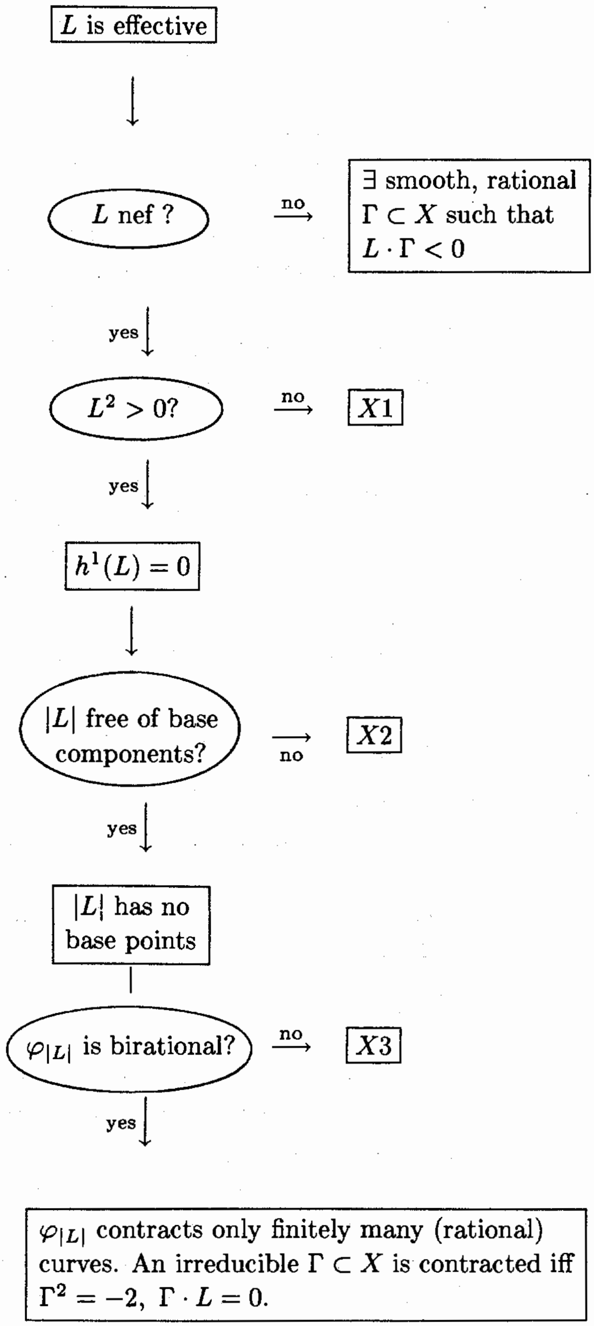
$X2$. Let $\Gamma \subset X$ be a smooth rational curve, $F \subset X$ smooth elliptic as above, and $\Gamma \cdot F = 1$. Consider $L := \mathscr{O}(kF + \Gamma)$, $k \geq 2$. We then have

$$h^0(L) = k+1, \qquad h^1(L) = 0,$$

and all divisors in $|L|$ are of the form $\Gamma + \sum_{i=1}^k F_i$ with $F_i \sim F$, so $\varphi_{|L|}$ has base-component Γ and maps X to a rational normal curve in $\mathbf{P}^k$.

$X3$. Let $D \subset X$ be a smooth hyperelliptic curve of genus $g \geq 2$, and let $L := \mathscr{O}(D)$. Then $\varphi_{|L|}$ is two-to-one, and every divisor in $|L|$ is hyperelliptic. If $(n-1)(g-1) > 1$, then the map $\varphi_{|nL|}$ is birational.

In a sense, these are the only cases of exceptional behavior. More precisely, let L be an effective line bundle on X. The properties of $|L|$ can be read off the following flow chart:



(2.2) A counterexample: nonpropagating g^1_4's. Let $\pi : X \to \mathbf{P}^2$ be a $K3$ surface of genus 2, i.e. a double cover of $\mathbf{P}^2$ branched along a nonsingular plane sextic curve $B \subset \mathbf{P}^2$. The line bundle of degree 2 given by $\pi^*\mathcal{O}_{\mathbf{P}^2}(1)$ is then just a special case of example $X3$. Instead we take $L := \pi^*\mathcal{O}_{\mathbf{P}^2}(3)$. We claim:

(i) $\varphi_{|L|}: X \to \mathbf{P}^{10}$ is an embedding.

(ii) There is a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\varphi_{|L|}} & \mathbf{P}^{10} \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle \mathrm{pr}} \\ \mathbf{P}^2 & \xrightarrow{v} & \mathbf{P}^9 \end{array}$$

where v is the Veronese embedding of $\mathbf{P}^2$ in $\mathbf{P}^9$ via the complete linear system $|\mathcal{O}_{\mathbf{P}^2}(3)|$, and pr is a linear projection.

(iii) Any hyperplane section of X which comes from $\mathbf{P}^9$ (i.e., factors through π) carries a 1-parameter family of g^1_4's.

(iv) The generic hyperplane section of X carries no g^1_4's, but does have a unique g^2_6.

The proofs are quite straightforward: let C be a nonsingular section of $|L|$. The sequence

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(L) \to \mathcal{O}_C(L) \to 0$$

gives rise to

$$0 \to H^0(\mathcal{O}_X) \to H^0(X, L) \to H^0(C, \omega_C) \to 0;$$

hence

$$h^0(X, L) = 1 + h^0(C, \omega_C) = 1 + g(C) = 11,$$

where the last step follows from

$$\begin{aligned} \deg(\omega_C) &= \deg(L|_C) = \deg(L) = \deg(\pi) \cdot \deg(\mathcal{O}(3)) \\ &= 2 \cdot 3^2 = 18 \Rightarrow g(C) = 10. \end{aligned}$$

We thus have a decomposition

$$H^0(X, L) \approx \pi^* H^0(\mathbf{P}^2, \mathcal{O}(3)) \oplus R,$$

where R is the 1-dimensional subspace of $H^0(X, L)$ consisting of sections vanishing on the ramification locus $\pi^{-1}(B) \subset X$. This proves claims (i) and (ii). If $C \subset X$ comes from $\mathbf{P}^9$ it is thus a double cover of a plane cubic $\pi(C) \subset \mathbf{P}^2$; the 1-parameter family of g^1_4's is just π^* of the 1-parameter family of g^1_2's on $\pi(C)$. For any other hyperplane section C, $\pi(C)$ is a plane sextic, whence the g^2_6; when $C \in \mathbf{P}(R)$ is the ramification curve, $\pi(C) = B$ is nonsingular by assumption, hence carries no g^1_4.

(2.3) A counterexample: g_d^1's which propagate but are not induced. Consider X as in (2.2), but now take $L := \pi^*\mathcal{O}_{\mathbf{P}^2}(4)$. A computation as above shows that for generic $C \in |L|$, $g(C) = 17$ and $\pi: X \to \mathbf{P}^2$ maps C (birationally) to a plane curve $\pi(C)$ of degree 8, hence with $(7\cdot 6)/2 - 17 = 4$ nodes. We see that the generic C has a g_8^2 as well as four g_6^1's; the g_8^2 is induced from a line bundle on X, but not the g_6^1's.

Let P_1, P_2, P_3, P_4 be the nodes of $\pi(C)$, and let $|Z|$ be the g_6^1 on C induced by the node P_1. If we fix a divisor $Z_0 \in |Z|$ consisting of distinct points, then there is some line l in $\mathbf{P}^2$ such that $l \cap \pi(C) = 2P_1 + \pi(Z_0)$; by choosing Z_0 appropriately we may assume that l does not contain P_i for $i \neq 1$, and that $\pi(Z_0)$ does not contain P_1. It is easily seen that the Brill-Noether number of $|Z|$ is $\rho = -7 < 0$.

Let us check that $h^0(\mathcal{O}_C(2Z)) = 3$; as we shall see in the next section, this is equivalent to the g_6^1 $|Z|$ being scheme-theoretically isolated in moduli. By duality, it suffices to check that $h^0(\mathcal{O}_C(\omega_C - 2Z)) = 7$.

Let $W \in |\omega_C - 2Z_0|$, so that $W + 2l \in |\omega_C|$. Then there is a plane curve D of degree 5 passing through P_1, P_2, P_3, and P_4 such that

$$D \cap \pi(C) = 2\sum_{i=1}^{4} P_i + W + 2Z_0.$$

Now $D \cap l \supset Z_0$ so that if l is not a component of D we have $5 = D \cdot l \geq \deg Z_0 = 6$, a contradiction. Thus, $D = D_1 \cup l$ with $\deg D_1 = 4$. Since

$$D_1 \cap \pi(C) = 2P_2 + 2P_3 + 2P_4 + W + Z_0,$$

a similar argument shows that $D_1 = D_2 \cup l$ with $\deg D_2 = 3$; moreover, $2P_1 \subset W$. Thus,

$$D_2 \cap \pi(C) = 2P_2 + 2P_3 + 2P_4 + (W - 2P_1).$$

Moreover, $D = D_2 \cup 2l$ passes through P_1, P_2, P_3 and P_4 so that D_2 must pass through P_2, P_3 and P_4.

We conclude that divisors in $|\omega_C - 2Z_0|$ are in one-to-one correspondence with plane cubics passing through P_2, P_3 and P_4. Since 3 points impose independent conditions on cubics (cf. Griffiths and Harris [5, p. 715]) we see that

$$h^0(\omega_C - 2Z) = 10 - 3 = 7,$$

as desired.

3. Linear systems on $K3$-sections propagate

(3.1) Theorem. *Let $X \subset \mathbf{P}^g$ be a $K3$ surface, and $C := X \cap H \subset \mathbf{P}^{g-1}$ a nonsingular hyperplane section of X. (C is canonically embedded in*

$\mathbf{P}^{g-1} \approx H$.) *If* C *has a* g_d^1 *which is scheme-theoretically isolated on* C*, then every nonsingular hyperplane section* C' *of* X *has a* g_d^1.

Let $\mathscr{J}_d^1$ denote the space of pairs consisting of a curve C and a g_d^1 on it, let $\mathscr{M}_d^1 \subset \mathscr{M}_g$ be the space of d-gonal curves, and for fixed $C \in \mathscr{M}_d^1$ let W_d^1 denote the fiber of $\mathscr{J}_d^1$ over C. We recall that the g_d^1 $|Z|$ on C is scheme-theoretically isolated if

$$T_{|Z|}W_d^1 = (0).$$

Equivalently, $\mathscr{J}_d^1$ must be transversal to the Jacobian of C. We have:

- $H^0(\omega - 2Z)$ injects into $H^0(\omega^2)$, and the image can be naturally identified with the conormal space at C to the local component of $\mathscr{M}_d^1$ corresponding to $(C, |Z|)$,
- $\dim \mathscr{J}_d^1 = 2g + 2d - 5$.

Putting these together, we see that the transversality is equivalent to

$$h^0(\omega - 2Z) + (2g + 2d - 5) = 3g - 3,$$

or

$$h^0(\omega - 2Z) = g - 2d + 2,$$

and by Serre duality, to

$$h^0(2Z) = 3.$$

Our theorem thus follows from the following more general statement:

(3.2) Theorem. *Let* $X \subset \mathbf{P}^g$ *be a* $K3$ *surface,* $C_0 := X \cap H_0$ *a nonsingular hyperplane section, and* $|Z|$ *a* g_d^r *on* C_0 *which is scheme-theoretically isolated on* C_0*, and satisfies*

$$h^0(C_0, \mathscr{O}(2Z)) = 2r + 1.$$

Then every nonsingular hyperplane section C *of* X *has a* g_d^r.

(3.3) Iterative construction. We construct a series of subvarieties $\mathscr{H}_i \subset (\mathbf{P}^g)^*$, $\mathscr{S}_i \subset S^d(X)$, and correspondences $\mathscr{I}_i, \mathscr{J}_i \subset (\mathbf{P}^g)^* \times S^d(X)$, as follows. Let $\mathscr{S}_0 := \{Z_0\}$ for some fixed divisor $Z_0 \in |Z|$ consisting of distinct points. Define inductively, for $i \geq 1$:

$$\mathscr{I}_i' := \{(Z, H) \in \mathscr{S}_{i-1} \times (\mathbf{P}^g)^* \mid H \supset \operatorname{span}(Z)\},$$

$\mathscr{I}_i$:=unique irreducible component of $\mathscr{I}_i'$ which dominates $\mathscr{S}_{i-1}$,

$$\mathscr{H}_i := \operatorname{pr}_2(\mathscr{I}_i) \subset (\mathbf{P}^g)^*,$$

$$\mathscr{J}_i := \left\{ (Z, H) \,\middle|\, \begin{array}{l} H \in \mathscr{H}_i, Z \in S^d C \text{ where } C := X \cap H \\ \exists Z' \in S^d C \text{ such that } (Z', H) \in \mathscr{I}_i \text{ and } Z \sim_C Z' \end{array} \right\},$$

where "$\sim_C$" means linear equivalence on C,

$$\mathscr{S}_i := \operatorname{pr}_1(\mathscr{J}_i) \subset S^d(X).$$

We note that for all $(Z, H) \in \mathscr{I}_i$,

$$h^0(X \cap H, \mathscr{O}(Z)) = r + 1.$$

This is an easy induction, based on the observation that the left-hand side depends, by the geometric version of Riemann-Roch, only on the position of the d-tuple Z in $\mathbf{P}^g$ and not on the choice of canonical curve through these points. Hence $\mathscr{J}_i$ is dominated by a $\mathbf{P}^r$-bundle over $\mathscr{I}_i$, so another easy induction shows that $\mathscr{J}_i$ is irreducible. (Actually, the same argument shows that $\mathscr{I}_i' = \mathscr{I}_i$ is already irreducible.)

Consider the following diagrams:

$$\begin{array}{ccc} & \mathscr{I}_i & \\ {\scriptstyle g-d+r}\swarrow & & \searrow \\ \mathscr{S}_{i-1} & & \mathscr{H}_i \end{array} \qquad\qquad \begin{array}{ccc} & \mathscr{J}_i & \\ \swarrow & & \searrow{\scriptstyle r} \\ \mathscr{S}_i & & \mathscr{H}_i \end{array}$$

What we know about them can be summarized as follows:

(1) All four maps are surjective.

(2) All fibers of $\mathrm{pr}_1 : \mathscr{I}_i \to \mathscr{S}_{i-1}$ are $(g - d + r)$-dimensional.

(3) All fibers of $\mathrm{pr}_2 : \mathscr{J}_i \to \mathscr{H}_i$ are at least r-dimensional; the fiber over H_0 has an irreducible component which is precisely r-dimensional, by our assumption that Z_0 is isolated.

The sequences $\mathscr{I}_i, \mathscr{J}_i, \mathscr{H}_i, \mathscr{S}_i$ stabilize for large i, and we let $\mathscr{I} = \mathscr{J}$, $\mathscr{H}$ and $\mathscr{S}$ denote the respective limits. From the diagrams we have:

$$\dim(\mathscr{S}) + g - d + r = \dim(\mathscr{I}) = \dim(\mathscr{J}) = \dim(\mathscr{H}) + r,$$

where the last step follows from (3) above together with the irreducibility of $\mathscr{J}$. Our theorem that $\dim(\mathscr{H}) = g$, is thus equivalent to $\dim(\mathscr{S}) = d$. In fact, we claim that already

$$\dim(\mathscr{S}_1) = d.$$

Indeed, $\mathrm{span}(Z_0)$ is a $\mathbf{P}^{d-r-1}$,, i.e. contained in a $(g - d + r)$-dimensional family of hyperplanes, i.e. $\dim(\mathscr{H}_1) = g - d + r$. Therefore,

$$\dim(\mathscr{J}_1) = g - d + 2r.$$

By the geometric version of Riemann-Roch, our assumption $h^0(C_0, \mathscr{O}(2Z)) = 2r + 1$ is equivalent to saying that for $Z_1 \neq Z_0$, $\mathrm{span}(Z_0, Z_1)$ is a $\mathbf{P}^{2d-2r-1}$. Hence the fibers of $\mathrm{pr}_1 : \mathscr{J}_1 \to \mathscr{S}_1$ have dimension $g - 2d + 2r$, so

$$\dim(\mathscr{S}_1) = (g - d + 2r) - (g - 2d + 2r) = d$$

as claimed. This proves Theorems (3.1) and (3.2).

4. Constancy of the Clifford index

Our main result in this section is a proof of Green's conjecture (1.1) for g_d^1's.

(4.1) Theorem. *Let C be a nonsingular curve of genus $g \geq 2$ on a $K3$ surface X, and suppose there is a g_d^1 $|Z|$ on C achieving the Clifford index, $\nu(C) = d-2$. Then $\nu(C) = \nu(\mathscr{O}_X(C))$.*

In view of the semicontinuity of the Clifford index, it will suffice to prove a particular case of conjecture (1.2): that there is a linear system on X whose restriction to C contains $|Z|$ and whose restriction to any $C' \in |C|$ has the same Clifford index as $|Z|$.

(4.2) Theorem. *Under the assumptions of* (4.1), *there is a divisor $D \subset X$ such that*

- $\nu(Z) = \nu(C) = \nu(\mathscr{O}_C(D))$.
- $h^0(\mathscr{O}_X(D)) \geq 2$, $h^0(\mathscr{O}_X(C-D)) \geq 1$, $\deg(\mathscr{O}_C(D)) \leq g-1$.
- *There is some $Z_0 \leq |Z|$, consisting of distinct points, such that $Z_0 \subset D \cap C$.*
- *For nonsingular $C' \in |C|$, $\nu(\mathscr{O}_{C'}(D)) = \nu(\mathscr{O}_C(D))$, $h^0(\mathscr{O}_{C'}(D)) \geq 2$ and $\deg(\mathscr{O}_{C'}(D)) \leq g-1$.*

There are two easy reduction steps in the proof of this theorem. First, we may assume that C is nonhyperelliptic (since the hyperelliptic case is covered by [14]), and hence that $\varphi_{|C|}$ is birational, and its restriction to C embeds C as a canonical curve. Second, notice that $|Z|$ is base-point-free and complete (else there would be a $g_{d'}^1$ or g_d^r with Clifford index $d'-2 < d-2$ or $d-2r < d-2$).

In §5, we will extend (4.2) to g_d^1's which do not necessarily achieve the Clifford index. We therefore state our hypotheses explicitly, so that our lemmas can be reused in §5. We assume only:

- C is a nonsingular nonhyperelliptic curve of genus $g \geq 2$.
- $|Z|$ is a complete base-point free g_d^1 on C, and a divisor $Z_0 \in |Z|$ has been chosen, consisting of distinct points none of which lies on any (of the countably many) rational curves on X.
- The Brill-Noether number $\rho(Z) = 2d-2-g$ is negative.

Our first lemma was inspired by work of Lazarsfeld and Reider.

(4.3) Lemma. *Under our hypotheses, there is a rank-2, nonsimple vector bundle $\mathscr{F} \to X$ with $c_1(\mathscr{F}) = [C]$ and $c_2(\mathscr{F}) = d$, and a section s of $\mathscr{F}$ with $(s) = Z_0$.*

Proof. We use a construction of Griffiths and Harris, Proposition (1.33) in [6]. This provides $\mathscr{F}$ and s with the required invariants; the condition needed is that any divisor in $|C|$ which passes through all-but-one points of Z_0 must pass through the remaining point. By surjectivity of

$$H^0(X, \mathcal{O}_X(C)) \twoheadrightarrow H^0(C, \omega_C)$$

we are reduced to the same condition for Z_0 and the canonical system, $|\omega_C|$. By Riemann-Roch, this is equivalent to our assumptions that $\dim |Z| > 1$ and that $|Z|$ is base-point-free.

We still need to check that $\mathscr{F}$ is nonsimple, i.e. that $h^0(\mathscr{F} \otimes \mathscr{F}^*) > 1$. But this is a straightforward computation (cf. Lazarsfeld [8] and Mukai [10]):

$$\begin{aligned}\chi(\mathscr{F} \otimes \mathscr{F}^*) &= c_1^2(\mathscr{F}) - 4c_2(\mathscr{F}) + 4\chi(\mathcal{O}_X) \\ &= 2g - 2 - 4d + 8 = -2\rho(Z) + 2 > 2,\end{aligned}$$

but since $\mathscr{F} \otimes \mathscr{F}^*$ is self dual,

$$\chi(\mathscr{F} \otimes \mathscr{F}^*) = 2h^0(\mathscr{F} \otimes \mathscr{F}^*) - h^1(\mathscr{F} \otimes \mathscr{F}^*)$$

so we conclude $h^0(\mathscr{F} \otimes \mathscr{F}^*) > 1$.

Remarks. (i) The bundle $\mathscr{F}$ in (4.3) is the dual of the one constructed by Lazarsfeld [8].

(ii) Reider's method [13] is as follows: the computation above shows that $c_1^2(\mathscr{F}) > 4c_2(\mathscr{F})$ exactly when $\rho(Z) < -3$. In that case, a theorem of Bogomolov [2] yields the conclusion in case (a) of Lemma (4.4) below.

(4.4) Lemma. *Let $\mathscr{F}$ be a nonsimple, rank-2 vector bundle on X. There exist line bundles L, M and a zero-dimensional subscheme $A \subset X$ such that $\mathscr{F}$ fits in an exact sequence*

$$0 \to L \to \mathscr{F} \xrightarrow{\pi} M \otimes \mathscr{I}_A \to 0$$

and either

(a) *$L \geq M$, or*

(b) *A is empty and the sequence splits, $\mathscr{F} \approx L \oplus M$.*

Proof. Since $\mathscr{F}$ is nonsimple, a standard argument shows the existence of an endomorphism $\varphi: \mathscr{F} \to \mathscr{F}$ which drops rank everywhere.[1] Let L, N be the kernel and image of φ respectively, and $M := N^{**}$, the double dual. Clearly, L and M are line bundles and $N = \mathscr{I}_A \otimes M$ for some zero-dimensional $A \subset X$.

The two cases arise as follows: if $\varphi^2 = 0$, then $N = \mathrm{im}(\varphi) \subset \ker(\varphi) = L$, so $L \otimes M^{-1} \approx L \otimes N^*$ has a section, and we are in case (a). Otherwise, φ

[1] If $\mathscr{F}$ is decomposable, take φ to be projection onto a summand. If $\mathscr{F}$ is indecomposable, let φ_0 be any automorphism of $\mathscr{F}$ which is not a multiple of the identity $1_{\mathscr{F}}$, and let λ be an eigenvalue of φ_0 at any point. Then $\varphi := \varphi_0 - \lambda 1_{\mathscr{F}}$ is not an automorphism, so by a theorem of Atiyah [1], it must be nilpotent; since $\varphi \neq 0$, it must drop rank everywhere.

must induce an isomorphism from N to its image in $\mathscr{F}$, thus splitting the sequence

$$0 \to L \to \mathscr{F} \to N \to 0.$$

Since $\mathscr{F}$ is locally free, N must be a line bundle, i.e., $A = \varnothing$ and we are in case (b).

(4.5) Corollary. *Under our hypotheses, there exist effective divisors D, Δ on X such that $C \sim D + \Delta$, $Z_0 \subset D \cap \Delta$, $D \cdot \Delta = d - \deg(A)$, and either*

(*Case* (a)) *$\Delta - D$ is effective, or*

(*Case* (b)) *D meets Δ transversally and $Z_0 = D \cap \Delta$.*

Proof. We apply (4.4) to (4.3). The section $s \in H^0(\mathscr{F})$ vanishes on the 0-dimensional locus Z_0, hence is not contained in the line-subbundle L. The projection $\pi(s)$ is therefore a nonzero section of $M \otimes \mathscr{I}_A$; let D be its 0-locus, so

$$M \approx \mathscr{O}_X(D), \qquad Z_0 \subset D.$$

In case (a) we take $\Delta = D + E$, where E is an effective divisor in $|L \otimes M^{-1}|$, so that $L \approx \mathscr{O}_X(\Delta)$, and we have

$$Z_0 \subset D = D \cap (D + E) = D \cap \Delta$$

and

$$d - \deg(A) = c_2(\mathscr{F}) - \deg(A) = D \cdot \Delta.$$

In case (b) we have a decomposition $s = s_L \oplus s_M$, so we define

$$D := (s_M), \qquad \Delta := (s_L).$$

Then Z_0 equals the intersection, which must be transversal since Z_0 consists of distinct points.

(4.6) Lemma. *Under our hypotheses, $\nu(\mathscr{O}_C(D)) \leq \nu(Z)$.*

Proof.

$$\begin{aligned} \nu(\mathscr{O}_C(D)) &= C \cdot D - 2h^0(\mathscr{O}_C(D)) + 2 \\ &\leq C \cdot D - 2h^0(\mathscr{O}_X(D)) + 2 \\ &\leq C \cdot D - (D \cdot D + 4) + 2 = \Delta \cdot D - 2 \\ &= d - \deg(A) - 2 \leq d - 2 = \nu(Z). \end{aligned}$$

The first inequality follows from the exact sequence

$$0 \to \mathscr{O}_X(-\Delta) \to \mathscr{O}_X(D) \to \mathscr{O}_C(D) \to 0,$$

and the second from Riemann-Roch for the line bundle $\mathscr{O}_X(D)$. q.e.d.

The proofs of Theorems (4.1) and (4.2) can now be completed: the extra hypothesis is that $\nu(Z)$ is minimal, so the inequality in (4.6) must be an equality. In particular, we must have:

(1) $H^0(\mathscr{O}_X(D)) \xrightarrow{\sim} H^0(\mathscr{O}_C(D))$ is an isomorphism;
(2) $H^1(\mathscr{O}_X(D)) = 0$
(3) $A = \varnothing$.

Combining (1) and (2) we get $H^1(D-C) = 0$. But then also $H^1(D-C') = 0$ for $C' \in |C|$, so we get an isomorphism:

$$H^0(\mathscr{O}_X(D)) \xrightarrow{\sim} H^0(\mathscr{O}_{C'}(D))$$

for nonsingular $C' \in |C|$, so finally

$$h^0(\mathscr{O}_{C'}(D)) = h^0(\mathscr{O}(_C(D))$$

as required.

5. Linear systems on $K3$-sections are contained in induced ones

(5.1) Theorem. *Let X be a $K3$ surface, $C \subset X$ a nonsingular, nonhyperelliptic curve, and $|Z|$ a complete, base-point-free g^1_d on C with $\rho(\mathscr{O}_C((Z)) < 0$. Then there is a line bundle $L \to X$ such that*

- $h^0(X, L) \geq 2$, $h^0(X, \mathscr{O}_X(C) \otimes L^{-1}) \geq 2$, $\deg(L \otimes \mathscr{O}_C) \leq g-1$.
- $\nu(\mathscr{O}_C \otimes L) \leq \nu(\mathscr{O}_C(Z))$.
- $\nu(\mathscr{O}_{C'} \otimes L) = \nu(\mathscr{O}_C \otimes L)$ *for nonsingular* $C' \in |C|$.
- *There are divisors $Z_0 \in |Z|$ (consisting of distinct points) and $D \in |L|$ such that $Z_0 \subset D \cap C$.*

For the proof we use the techniques of §4, with one new idea. The problem is that even after we have manufactured the splitting $C \sim D + \Delta$, we are not done: the inequalities in (4.6) may not be equalities, so $H^0(\mathscr{O}_C(D))$ may be bigger than $H^0(\mathscr{O}_X(D))$, and no conclusion can be made about $\nu(\mathscr{O}_{C'}(D))$.

We thus introduce a definition: a line bundle $L = \mathscr{O}_X(D)$ is *adapted to* $|C|$ if

(1) $h^0(\mathscr{O}_X(D)) \geq 2$, $h^0(\mathscr{O}_X(C-D)) \geq 2$, and
(2) $h^0(\mathscr{O}_{C'}(D))$ is independent of the nonsingular $C' \in |C|$.

The theorem can thus be rephrased:

(5.1′) Theorem. *Let X be a $K3$ surface, $C \subset X$ a nonsingular, nonhyperelliptic curve, and $|Z|$ a complete, base-point-free g^1_d on C with $\rho(\mathscr{O}_C(Z)) < 0$. Then there is a line bundle $L \to X$ adapted to $|C|$ such that*

- $\nu(L \otimes \mathscr{O}_C) \leq \nu(Z)$.
- *For some divisors $Z_0 \in |Z|$ (distinct points) and $D \in |L|$, $Z_0 \subset D \cap C$.*

(5.2) Lemma. *$L = \mathscr{O}_X(D)$ is adapted to $|C|$ if*
(1) $h^0(\mathscr{O}_X(D)) \geq 2$, $h^0(\mathscr{O}_X(C-D)) \geq 2$, *and*
(2′) *Either* $h^1(\mathscr{O}_X(D)) = 0$ *or* $h^1(\mathscr{O}_X(C-D)) = 0$.

Proof. The sheaf sequence

$$0 \to \mathscr{O}_X(D - C') \to \mathscr{O}_X(D) \to \mathscr{O}_{C'}(D) \to 0$$

gives

$$0 \to H^0(\mathscr{O}_{X'}D - C') \to H^0(\mathscr{O}_X(D)) \to H^0(\mathscr{O}_{C'}(D)) \\ \to H^1(\mathscr{O}_X(D - C')) \xrightarrow{\alpha} H^1(\mathscr{O}_X(D)).$$

We note that

$$h^i(\mathscr{O}_X, D - C') = h^i(\mathscr{O}_X, D - C)$$

is independent of C'. Hence $h^0(\mathscr{O}_{C'}(D))$ is determined by $\operatorname{rank}(\alpha)$; the alternatives in (2′) assure $\operatorname{rank}(\alpha) = 0$. (Note that $h^1(\mathscr{O}_X(C - D)) = h^1(\mathscr{O}_X(D - C)) = h^1(\mathscr{O}_X(D - C'))$.)

(5.3) Proposition. *Let D be a divisor on X such that $h^0(\mathscr{O}_X(D)) \geq 2$ and $h^0(\mathscr{O}_X(C - 2D)) \geq 1$. Then there is a divisor $\tilde{D}$ on X such that*

(i) *$\mathscr{O}_X(\tilde{D})$ is adapted to $|C|$.*

(ii) *$h^0(\mathscr{O}_X(C - 2\tilde{D})) \geq 1$.*

(iii) *$\tilde{D} \cdot (C - \tilde{D}) \leq D \cdot (C - D)$.*

(iv) *For some Γ_0 which is either empty or a smooth rational curve, $D - \tilde{D} + \Gamma_0$ is an effective divisor whose support is a union of smooth rational curves.*

Proof. Let E be an effective divisor in the linear system $|C - 2D|$. We apply (2.1) to $\mathscr{O}_X(D)$.

Suppose first that D is nef. If $D^2 > 0$ then $h^1(\mathscr{O}_X(D)) = 0$ and $\mathscr{O}_X(D)$ is adapted to $|C|$ by Lemma (5.2); set $\tilde{D} := D$. Otherwise, $D^2 = 0$ and $\mathscr{O}_X(D)$ has the type of example $X1$, that is, $D \sim kF$ for some smooth elliptic curve F. If $k = 1$, then $h^1(\mathscr{O}_X(D)) = 0$ so $\mathscr{O}_X(D)$ is still adapted to $|C|$ and we may set $\tilde{D} = D$.

Thus, we may assume $D \sim kF$ with $k \geq 2$. We now apply (2.1) to $\mathscr{O}_X(D + E)$. If $D + E$ is not nef, let Γ_0 be a smooth rational curve such that $(D + E) \cdot \Gamma_0 < 0$, and let $\tilde{D} := D + \Gamma_0 \sim kF + \Gamma_0$. We claim that $\tilde{D}$ is nef: the only curve which could possibly have negative intersection number with $\tilde{D}$ is Γ_0, but

$$F \cdot \Gamma_0 = \frac{1}{k} D \cdot \Gamma_0 = \frac{1}{k}(C \cdot \Gamma_0 - (D + E) \circ \Gamma_0) \geq -\frac{1}{k}(D + E) \cdot \Gamma_0 > 0,$$

so that

$$\tilde{D} \cdot \Gamma_0 = kF \cdot \Gamma_0 - 2 \geq k - 2 \geq 0.$$

Thus $\tilde{D}$ is nef: moreover, $\tilde{D}^2 = (kF + \Gamma_0)^2 = 2kF \cdot \Gamma_0 - 2 \geq 2k - 2 \geq 2$, so that $h^1(\mathscr{O}_X(\tilde{D})) = 0$ by (2.1). Hence $\mathscr{O}_X(\tilde{D})$ is adapted to $|C|$.

We must check the other properties claimed for $\tilde{D}$ in this case. Since

$$E \cdot \Gamma_0 = (D + E) \cdot \Gamma_0 - kF \cdot \Gamma_0 \leq (D + E) \cdot \Gamma_0 - k < -k \leq -2,$$

we have $E - \Gamma_0$ effective. Furthermore,

$$(E - \Gamma_0) \cdot \Gamma_0 = E \cdot \Gamma_0 + 2 < 0,$$

so that $E - 2\Gamma_0$ is effective as well. Thus,

$$h^0(\mathscr{O}_X(C - 2\tilde{D})) = h^0(\mathscr{O}_X(E - 2\Gamma_0)) \geq 1,$$

verifying property (ii). Property (iv) is clear from the definition of $\tilde{D}$; to check property (iii), we compute

$$\begin{aligned} \tilde{D} \cdot (C - \tilde{D}) &= D \cdot (C - D) + \Gamma_0 \cdot (C - 2D) - \Gamma_0^2 \\ &= \tilde{D} \cdot (C - D) + \Gamma_0 \cdot E + 2 \leq D \cdot (C - D). \end{aligned}$$

To complete the proof in the case that D is nef, we may thus assume $D \sim kF$ with $k \geq 2$ and $D + E$ is nef. If $(D + E)^2 > 0$, then by (2.1),

$$h^1(\mathscr{O}_X(C - D)) = h^1(\mathscr{O}_X(D + E)) = 0$$

so that $\mathscr{O}_X(D)$ is once again adapted to $|C|$ by (5.2), and we may set $\tilde{D} := D$. Otherwise, $D + E \sim lG$ for some smooth elliptic curve G, and every divisor in $|D+E|$ has the form $G_1 + \cdots + G_l$ for certain $G_i \in |G|$. Since $kF + E \in |D+E|$, we must in fact have $|F| = |G|$. But then $C \sim (k + l)F$ so that $C^2 = 0$, a contradiction.

To prove the proposition in general, we use induction on the number of base components of $|D|$, counted with multiplicity. If $|D|$ has no base components then D is nef and we are finished. If $|D|$ has m base components, we may assume that D is not nef (else we are finished as above) and let Γ be a smooth rational curve with $D \cdot \Gamma < 0$. Then Γ is a base component of $|D|$, and $|D - \Gamma|$ has $m - 1$ base components. By inductive hypothesis, there is a $\tilde{D}$ adapted to $|C|$ with $h^0(\mathscr{O}_X(C - 2\tilde{D})) \geq 1$ such that $\tilde{D} \cdot (C - \tilde{D}) \leq (D - \Gamma) \cdot (C - D + \Gamma)$ and $(D - \Gamma) - \tilde{D} + \Gamma_0$ is effective and supported on rational curves for some Γ_0. Since $D - \tilde{D} + \Gamma_0 = ((D - \Gamma) - \tilde{D} + \Gamma_0) + \Gamma$, it suffices to show that

$$(D - \Gamma) \cdot (C - D + \Gamma) \leq D \cdot (C - D),$$

i.e., since $(D - \Gamma) \cdot (C - D + \Gamma) = D \cdot (C - D) - \Gamma \cdot E + 2$ it suffices to show that $\Gamma \cdot E \geq 2$. But $\Gamma \cdot D \leq -1$ so that

$$\Gamma \cdot E = \Gamma \cdot C - 2\Gamma \cdot D \geq -2\Gamma \cdot D \geq 2. \quad \text{q.e.d.}$$

We can now complete the proof of (5.1). We choose $Z_0 \in |Z|$ as in §4, consisting of distinct points not on any nonsingular rational curve in X. We apply (4.5) to obtain D, Δ, with $D \cap \Delta \supset Z_0$.

In case (a) of (4.5), we use Proposition (5.3) to replace D by $\tilde{D}$ which is adapted to $|C|$, with

$$\tilde{D} \cdot (C - \tilde{D}) \leq D \cdot (C - D) = D \cdot \Delta.$$

Since $D - \tilde{D} + \Gamma_0$ is supported on rational curves, it does not meet Z_0, so $Z_0 \subset D \Rightarrow Z_0 \subset \tilde{D}$. We now apply Lemma (4.6) to $\tilde{D}$, concluding that $\nu(\mathscr{O}_C(\tilde{D})) \leq \nu(Z)$. We may thus take $L := \mathscr{O}_X(\tilde{D})$.

In case (b) of (4.5), we simply take $L := \mathscr{O}_X(D)$. We claim:

$$h^1(\mathscr{O}_X(D)) = h^1(\mathscr{O}_X(\Delta)) = 0,$$
$$h^0(\mathscr{O}_X(D)) \geq 2, \qquad h^0(\mathscr{O}_X(\Delta)) \geq 2.$$

By symmetry, it suffices to check this for D. We use the results of (2.1):

If D is not nef: there is a smooth, rational Γ such that $D \cdot \Gamma < 0$, so $D_0 := D - \Gamma$ is effective. We have

$$\Gamma \cdot \Delta = \Gamma \cdot (C - D) = \Gamma \cdot C - \Gamma \cdot D > 0 - 0 = 0,$$

so $Z_0 = D \cap \Delta \supset \Gamma \cap \Delta$ must contain a point of Γ, a contradiction.

If $D^2 > 0$ then $h^1(\mathscr{O}_X(D)) = 0$, $h^0(\mathscr{O}_X(D)) \geq 2$ and we are done. By (2.1), the only remaining case is $X1$:

$$D \sim kF, \qquad F \text{ nonsingular elliptic}, k \geq 1,$$

and then

$$h^0(L) = k + 1, \qquad h^1(L) = k - 1.$$

We claim that $k = 1$. Indeed,

$$D \cdot C = D \cdot (C - D) = D \cdot \Delta = d,$$

so $Z_0 = D \cap C$, hence

$$2 = h^0(\mathscr{O}_C(Z)) = h^0(\mathscr{O}_C(D)) \geq h^0(\mathscr{O}_X(D)) \geq 2$$

so

$$k + 1 = h^0(\mathscr{O}_X(D)) = 2$$

as required.

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